13. Financial fragility, mean-field interaction and macroeconomic dynamics: a stochastic model

Corrado Di Guilmi, Mauro Gallegati and Simone Landini

13.1. INTRODUCTION

In recent decades, a considerable stream of research, following the complexity approach (Rosser, 2004), has developed a series of models that import concepts and tools from hard sciences to economics. They represent an attempt to identify an alternative framework to the representative agent hypothesis and to its underlying simplified solution to the aggregation problem (Kirman, 1992). Theoretical research has moved in two main directions: first, the development of agent-based models, solved by means of computer simulations (Axtell et al., 1996; Axelrod, 1997); second, formulations of stochastic frameworks for the aggregation of micro-variables (Aoki, 1996, 2002; Aoki and Yoshikawa, 2006). Despite major advances, the problem of how to analytically determine an explicit and closed solution for this class of models, as commonly performed in traditional economic analysis, remains substantially unsolved. Indeed, a considerable methodological problem arises, since the introduction of heterogeneous (in time and space) agents prevents the common tools of the economist being used. To overcome this obstacle, different aggregation methods have been proposed. Yet they are still waiting for an application that, using explicit functions and formulations, succeeds in proving their usefulness.

With this work we seek to fill this gap. Starting from the pioneering works of Aoki, we develop a model of financial fragility, along the lines of Greenwald and Stiglitz (1993) and Delli Gatti et al. (2005), in a dynamic stochastic framework, where the interaction of heterogeneous agents in conditions of uncertainty gives rise to economic fluctuations.

Delli Gatti et al. (2005) re-built the model of financial fragility presented in Greenwald and Stiglitz (1990, 1993) with the introduction of indirect interaction of heterogeneous agents and solved it numerically. As amply shown in that work, a complex system structure appears to be particularly suitable to model a financial fragility approach in which, given the existence of different financial structures, the distribution of agents matters (Forni and Lippi, 1997). The financial instability hypothesis, since its first formulation (Minsky, 1963), considers firms heterogeneous as regards the proportions of short-term debt on total sources of finance. The quantitative outcomes of the model replicate a number of stylized facts, strengthening the idea that the economy would be better represented as a complex dynamic system rather than a mere sum of identical perfectly informed agents.

From a methodological point of view, the recent works of Masanao Aoki introduced into economics the concept of mean-field interaction, by means of which analytical modelling of interaction among heterogeneous agents becomes possible. Mean-field interaction can be defined as the average interaction model that approximates the interactions among agents that could not otherwise be analytically treated (Opper and Saad, 2001). Agents are clustered in a set of states, with regard to one particular feature (the micro-state, *e.g.* the level of production for each firm) that determines the characteristics and the evolution of the aggregate (the macro-state, *e.g.* the total level of output). The focus is not on the single agent, but on the number or proportion of agents that occupy a certain state at a certain time. These levels are governed by a stochastic law which also defines the functional of the probability distributions of aggregate variables and, if they exist, their equilibrium distributions. The stochastic aggregation is then made effective through master equation techniques, that allow us to describe the dynamics of probability flows and hence determine the aggregate effects of underlying fluctuations in agents' configurations, and therefore the consequent changes at macro-level.

The main result of this chapter is the development and use of a framework for aggregating heterogeneous agents that proves capable of originating fluctuations of total production around a long path trend. Consistent with the inspiring financial fragility models, aggregate output is an inverse function of the system's degree of financial fragility, but with a different and more consistent microfoundation, given that objective functions differ for firms that have a dissimilar financial condition. Unlike the work of Delli Gatti et al. (2005), the interaction of different units, which gives rise to the complex dynamics, is analytically modelled by means of innovative stochastic aggregation tools. Indeed, with respect to Aoki's results, we manage to find a closed-form solution for a model with interacting and heterogeneous agents. In particular, we present an explicit steady-state solution for the master equation that leads to the analytical identification of an ordinary differential equation to describe trend dynamics and of a probability distribution function for fluctuations.

The structure of the work is the following: first, we specify the hypothesis for the stochastic structure of the system (Section 13.2) and for firms that comprise it (Section 13.3). In section 13.4, we develop the framework, setting the dynamic instruments required for aggregation, and solve the model, determining the two equations that drive production trends and business cycles. Some results of the model's computer simulations are presented in section 13.5, followed by concluding remarks in section 13.6.

13.2. THE PROBABILISTIC STRUCTURE

We set up a model in continuous time for a system of heterogeneous and interacting agents, partitioned into groups or states. In this section we state the hypothesis regarding the definition and structure of the states. Our system is structured into two states. Along time, a single firm can be in one of them, depending on its financial soundness, quantified by its equity ratio, i.e. the ratio of net worth to total assets. We use 0 to identify the state of firms with an equity ratio $a_i(t)$ no lower than the threshold $\overline{a}$, and 1 for the state of firms which have an equity ratio lower than the threshold and are thus exposed to the risk of demise. The twofold partition permits us to isolate the effect of the expected bankruptcy costs on aggregate dynamics. For the analytical treatment, all the a_i are approximated by the two variables a^1 and a^0. It is thus feasible to obtain the mean-field approximation of interactions among agents. More precisely, $a^j : j=0, 1$ can be regarded as a statistic of all the equity ratios for each state.

The choice to let the system work in continuous time is due, on the one hand, to analytical reasons, since it gives the option to use a set of analytical tools that cannot be employed in discrete time. Continuous time also appears to be more consistent with a complexity approach (Hinich et al., 2006).

The economy is populated by a fixed number of firms N, each indexed by i. Let us define the occupation numbers N^j for $j=0, 1$ as the number of firms, respectively, in state 0 and state 1 and n^j as their percentage. By assumption, the dynamics of these occupation numbers follow a continuous-time jump Markov process. *A priori* probability of being in state 1 is indicated by η, such that:

$$p(1)=\eta \Leftrightarrow p(0)=1-\eta$$

Firms exit from the system only for bankruptcy. In order to maintain constant the number of firms N, we assume that each bankrupted firm is immediately substituted by a new one. New firms, by assumption, enter the system in state 1 (that is with a low equity ratio). Therefore failures of firms do not modify N^1.

In order to model the probabilistic flows of firms from one state to another, we have to properly define transition probabilities and transition rates. The former are the probabilities of a single firm switching from one state to the other in a given instant. Namely, ζ is the transition probability of moving from state 0 to 1 (firms whose financial position is deteriorating, with equity ratio that becomes lower than $\overline{a}$) and ι indicates the probability of the opposite transition (firms whose equity ratio improves to become greater than $\overline{a}$). Moreover, we use μ to indicate the probability of bankruptcy.

The particular configuration assumed by agents, jointly determined by these probabilities, thus gives rise to a particular macro-state at aggregate level. This macro-state can be identified by the occupation number. The set of macro-states is therefore represented by all the possible occupation numbers N^j. Given that N does not change, it is enough to observe one of the states. Therefore, we can take as state variable the numbers of firms in 1, defining N_k as the number of firms occupying it in a given instant, with $0 \leq N_k \leq N$. Once a particular macro-state $N_k(t)$ is verified, it can be modified by a 'birth' from state 0 to state 1, and by a 'death' from state 1 to state 0. Thus, in order to describe the dynamic behaviour of the system, we need to specify the aggregate probability of observing, in a unit of time, a 'jump' of an agent from one state to another, and a consequent variation in occupation numbers, given the particular starting macro-state. This measure of probability is quantified by transition rates. We specify a transition rate for entries *into* state 1, λ, and a transition rate for exits *out* from the same state, γ, according to the following formulation:

$$\begin{aligned} \lambda &= \zeta(1-\eta) \\ \gamma &= \iota\eta \end{aligned} \qquad (13.1)$$

Therefore, the transition rate is given by the probability of a firm switching from one state to another, weighted by the probability of being in the starting state. Multiplying the result by the actual occupation numbers, we obtain the probability of observing this transition in a unit of time:

$$\begin{aligned} b(N_k) &= P(N_{k+1}(t') \,|\, N_k(t)) = \lambda(N - N_k) \\ d(N_k) &= P(N_{k-1}(t') \,|\, N_k(t)) = \gamma(N_k) \end{aligned} \qquad (13.2)$$

where b and d indicate, respectively, 'births' (transition to N_{k+1}) and 'deaths' (transition to N_{k-1}) of the stochastic process and t'^{+}.

The probability of bankruptcy for a distressed firm is indicated with μ. It represents a key variable in determining the evolution path and the dynamics of fluctuation in the economic system, as detailed in the following section. The rate of exit from the system is then quantified by $\mu\eta$, which, given the hypothesis, also represents the rate of entry into the system.

The mechanism of the system[1] is depicted in Figure 13.1.

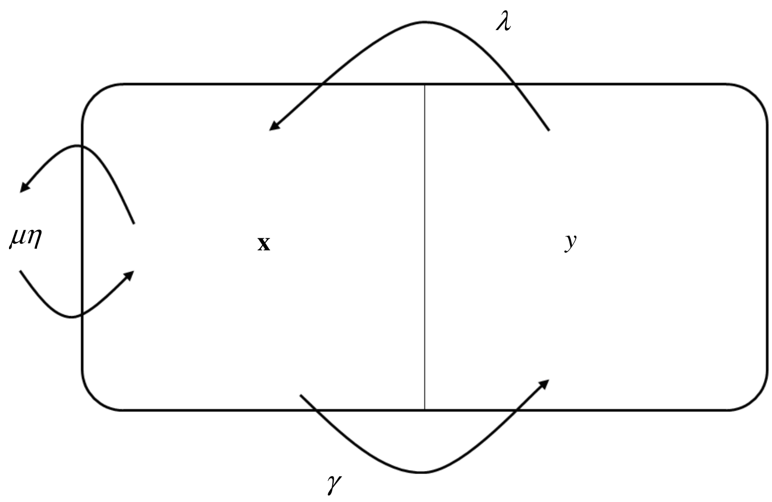

Figure 13.1. System structure

13.3. THE FIRMS

13.3.1 Hypothesis

Firms' behaviour is adapted from the original models of Greenwald and Stiglitz (1990, 1993), as modified by Delli Gatti et al. (2005). The adjustments are mainly due to the convenience of keeping the computational mechanism as simple as possible in order to highlight the aggregation problem and the proposed solution. At the same time, the framework maintains consistency with the original bankruptcy costs approach and a degree of heterogeneity among firms that is suited to the stochastic dynamics described in the previous section.

In Delli Gatti et al. (2005) the effect of firms' financial fragility on aggregate production is made effective through the credit market mechanism, that reduces the supply of credit when a firm fails. In this work, financial variables and production influence each other through feedback effects from the system to the agents and *vice versa*. Indeed, aggregate production is determined by the occupation numbers, and the probability of observing a change in state is itself influenced by the general financial situation of the economy. Interaction is not direct, but is represented as a mean-field interaction, that is to say, mediated at a meso-level of aggregation.

The economy is supply-driven: each of the N firms sells all the output that it (optimally) decides to produce. The produced good is perishable. Firms are identical within each state. Firms employ only one factor in the production process, indicated as the physical capital k, that does not depreciate. Unlike Delli Gatti et al. (2005), the total number of firms N does not change over time. Firms are identical within each state. In what follows the apex refers to the state while the firm index is in subscript.

According to these hypotheses the aggregate production function of the economy is given by:

$$Y(t) = N^1 y^1 + N^0 y^0 \tag{13.3}$$

where y^1 and y^0 indicate the optimal level of production for firms in the two states. The significance and dynamic structure of the occupation numbers N^1 and N^0 are detailed above. In the following we define the mechanism for determining firms' production levels. The production function of a generic firm i is:

$$q_i(t) = 2\left[k_i(t)\right]^{1/2} \tag{13.4}$$

where q is the physical output. Therefore, the demand function for capital of a single firm will be:

$$k_i(t) = \frac{1}{2}\left[q_i(t)\right]^2 \tag{13.5}$$

Basing on equation (13.5), firms adjust their capital stock to optimize their production level, investing an amount equal to $I_i(t)$, which can be either positive or negative, since a firm that reduces its production also has to diminish its physical stock.[2]

As assumed by Greenwald and Stiglitz (1993), firms are fully rationed on the equity market, such that they can finance their investments only with internal sources (A_i) or with loans (B_i). Equity is given by the profits that firms retain over time such that the law of motion of A_i is given by:

$$A(t') = A(t) + \pi(t')$$

Variation in the stock of debt $(\delta B_i(t'))$ in a period will then be:

$$\delta B_i(t') = I_i(t') - A_i(t) \tag{13.6}$$

In order to maintain balance sheet identity, we have $K_i(t) = A_i(t) + B_i(t)$. Under the assumption that firms and banks sign long-term contractual relationships (with no set date of repayment), at each t debt commitments in real terms for the i^{th} firm are $rB_i(t)$, where r is the real interest rate. Supposing that the market for capital is always in equilibrium (such that the real return on net worth equals the interest rate r), each firm incurs financing costs equal to:

$$r\left[B_i(t) + A_i(t)\right] = rK_i(t) \tag{13.7}$$

We can assume that the interest rate is constant over time and the same for all firms since demand for credit is endogenously limited within the model and cannot grow indefinitely.

At the moment of selling, each firm faces an iid idiosyncratic real shock in price. The i^{th} individual selling price is the random outcome of a market process around the average market price $P(t)$, according to the law:

$$P_i(t) = \tilde{u}_i(t)P(t) \tag{13.8}$$

The random variable $\tilde{u}_i(t)$ has uniform distribution with $E(\tilde{u}) = 1$, where E is the mathematical average. Without loss of generality its support can be fixed in the interval $[0.75; 1.25]$. Choice of the range for $\tilde{u}$ does not affect probabilities, given the normalization procedures detailed below. Once a firm fails, it faces bankruptcy costs $C_i(t)$ growing with firm size, quantified by sales volume, and given by:

$$C_i(t) = c\left[P_i(t)q(t)_i\right]^2 = c\left[P(t)\tilde{u}_i(t)q_i(t)\right]^2 : 0 < c < 1 \tag{13.9}$$

Greenwald and Stiglitz (1993) use a linear function but their results, as they state, hold for any function convex in q. The introduction of bankruptcy costs may be justified by the behaviour of managers, who act to avoid failure for two main reasons (Greenwald and Stiglitz, 1990). First, given the impossibility of identifying the cause of default in bad luck or in managers' incompetence, bankruptcy may generate stigmatization of managers' behaviour and, as a consequence, a reduction in their future earnings. Second, the imposition of a penalty for failure constitutes an incentive, since managers' contracts cannot state a participation in losses. As a further consequence, bankruptcy costs grow with size, since the number of employed managers is determined by the scale of operation.

13.3.1.1. Transition probabilities

A firm fails when its own capital A_i goes to 0. Given that the only exogenous variable in the model is the stochastic price shock $\tilde{u}_i(t)$, a bankruptcy condition may be defined as a function of the price shock:

$$\tilde{u}(t') \leq \frac{P(t)}{P(t')}\left[\frac{rK_i(t)}{q_i(t)} - a^1(t)\frac{k_i(t)}{P(t')q_i(t)}\right] \equiv \overline{u}_i(t') \qquad (13.10)$$

Substituting equation (13.5) into the above expression and, without loss of generality, normalizing reference price $P(t)=P(t')$ to 1, the r.h.s. of equation (13.10) becomes:

$$\overline{u}(t') \equiv \frac{q^1(t)}{2}\left[r - a^1(t)\right] \qquad (13.11)$$

Having the random variable $\tilde{u}$ support [0.75; 1.25], we can quantify the critical thresholds of shock prices for having bankruptcy as:

$$\begin{cases} \overline{u}_i(t) = 0.75 & \text{if } \overline{u} < 0.75 \\ \overline{u}_i(t) \leq \tilde{u}_i(t) & \text{if } 0.75 \leq \overline{u} \leq 1.25 \\ \overline{u}_i(t) = 1.25 & \text{if } \overline{u} > 1.25 \end{cases} \qquad (13.12)$$

It is then possible to indicate the probability of failure $\mu(t)$ for a firm as a function of $\tilde{u}(t)$:

$$\mu(t) = F(\overline{u}(t)) = \frac{\overline{u}(t) - 0.75}{0.5} = 2\overline{u}(t) - 1.5 \qquad (13.13)$$

Equation (13.13) permits endogenous determination of the equity ratio threshold $\overline{a}$. It can be interpreted as the minimum equity ratio which ensures a firm's survival, that is for which the probability of bankruptcy is equal to zero[3], and hence can be expressed as:

$$\overline{a}(t') = r - \frac{1.5}{q^1(t)} \qquad (13.14)$$

Analogously, the transition probabilities ζ and ι can be expressed as dependent variables of the price shock $\tilde{u}_i(t)$, with the opportune critical values, indicated, respectively as $\overline{u}_\zeta(t)$ and $\overline{u}_\iota(t)$:

$$\tilde{u}_i(t) \leq \frac{q^0(t)}{2}\left(r+\bar{a}(t)-a^0(t)\right) \equiv \bar{u}_\zeta(t)$$
$$\tilde{u}_i(t) > \frac{q^1(t)}{2}\left(r+\bar{a}(t)-a^1(t)\right) \equiv \bar{u}_\iota(t) \tag{13.15}$$

The range of variation of the two thresholds is truncated as in (eq:baru), explicit formulation for transition probabilities can now be straightforwardly calculated:

$$\zeta(t) = p(\tilde{u}(t) \leq \bar{u}_\zeta(t)) = 2\bar{u}_\zeta(t) - 1.5 \tag{13.16}$$

$$\iota(t) = 1 - p(\tilde{u}(t) \leq \bar{u}_\iota(t)) = -2\bar{u}_\iota(t) + 2.5 \tag{13.17}$$

13.3.2. Firms' objective function and aggregate production

Each firm acts to maximize its profit function:

$$\max_{q_i(t)} F(q_i(t)) = E\left\{\left[P(t)u_i(t')q_i(t)\right] - rK_i(t) - C_i(t)\mu(t')\right\} \tag{13.18}$$

We suppose that firms allow for the present level of failure probability, hence $E[\mu(t')] = \mu(t)$, and that $E[P(t')] = P(t') = P(t) = 1$, without loss of generality. The first order condition is:

$$1 - rq_i(t) - 2cq_i(t)\mu(t) = 0$$

Consequently, there are two different optimal levels of production, one for firms in state 1 and one for firms in state 0, respectively:

$$q^{1*} = (r+2c\mu)^{-1}$$
$$q^{0*} = r^{-1} \tag{13.19}$$

given that $\mu = 0$ for firms in state 0. The aggregate output of the system given by equation (13.3) can now be expressed as:

$$Y(t) = \frac{N^1(t)}{r+2c\mu(t)} + \frac{N^0(t)}{r} \tag{13.20}$$

As can be seen, fluctuations have two main sources: the first concerns the failure probability μ and the second is related to the dynamics of the occupation numbers. To highlight the impact of financial fragility, we may express equation (13.20) in the following way:

$$Y(t) = \frac{N}{r} - \xi(t)N^1(t) \tag{13.21}$$

where $\xi(t) = \frac{1}{r}\left(1 + \frac{r}{2c\mu(t)}\right)^{-1}$. The second member of the r.h.s. in equation (13.21) is the product of the two independent random variables $\xi(t)$ and N^1. The latter can be considered a macroeconomic indicator for the financial fragility of the system; it is weighted by ξ, which is a function of probability μ that, depending upon a^1, reflects financial conditions of firms in state 1. Therefore the dynamics of aggregate production is negatively correlated to the micro- and macro-level of financial distress in the economy.

13.4. STOCHASTIC DYNAMICS

13.4.1. Master equation

In order to describe the dynamics of aggregate production, we need to specify the stochastic evolution of micro-states, which determines the evolution of occupation number probabilities. For this task we need to set up and solve the master equation. A master (or Chapman–Kolmogorov) equation can be defined as a first-order differential equation that describes the dynamics of the probability of a system occupying each of a determinate set of macro-states. It 'describes the time evolution of the probability distribution of states, not the time evolution of the states themselves' (Aoki, 2002, p. 7). Hence it permits us to quantify the variation of probability fluxes in a small interval of time for continuous-time Markov processes. According to this definition, the probability distribution of having N_k firms in state 1 in a given instant will follow this scheme:

$$\frac{dP(N_k)}{dt} = \text{(inflows of probability fluxes into 1)} - \text{(outflows of probability fluxes out of 1)}.$$

To specify the dynamics of joint probabilities and thereby the stochastic evolution of the system, we make use of the following master equation[4]:

$$\begin{aligned}\tfrac{dP(N_k)}{dt} &= b(N_{k-1})P(N_{k-1}) + d(N_{k+1})P(N_{k+1}) + \\ &-\{[(b(N_k) + d(N_k))P(N_k)]\}\end{aligned} \tag{13.22}$$

with boundary conditions:

$$\begin{cases} P(N, t) = b(N^1)P(N^1 - 1, t) + d(N)P(N, t) \\ P(0, t) = b(1)P(1, t) + d(0)P(0, t) \end{cases} \tag{13.23}$$

These conditions ensure that the distribution functions only consider meaningful values, that is to say $N^1 \in [0; N]$.

A master equation is a balance flows equation that sums the probability of reaching the number N_k of firms in state 1 with a 'death' from N_{k+1} or with a 'birth' from N_{k-1}, less the probability of already being in the macro-state N_k and observing a transition to or from micro-state 1. It fully describes the dynamics of our system. Below we show that, by means of this analytical instrument, it is possible: first, to determine the endogenous steady-state value of the probability η that we have so far treated as exogenous; second, to describe the trend and fluctuations of occupation numbers and hence of aggregate production.

13.4.2. Stationary points and equilibrium probability

In this section we define an endogenous formulation for the probability η of being in state 1. By definition, steady state implies that the probability of in-fluxes equates the probability of out-fluxes for each state. This condition is known as detailed balance and is obtained by equating the master equation to 0. Analytical details are provided in Appendix A. Stationary probability can be derived by applying Brook's lemma (Brook, 1964) that quantifies the equilibrium probability $P^e(N_k)$ which ensures that detailed balance holds for each pair of macro-states:

$$P^e(N_k) = P^e(N(0)) \left(\frac{\iota}{\zeta} \right)^{N_k} \binom{N}{N_k} \prod_{h=1}^{H} \frac{\eta(N - N_h)}{(1 - \eta(N_h))} \tag{13.24}$$

Thanks to Hammersley and Clifford's theorem (Clifford, 1990), it is possible to treat the Markovian space as a Gibbs space, expressing $P^e(N_k)$ in Gibbs form. Then, by following Landini (2005, ch. 5) the probability η of a firm being in state 1 can be expressed as a function of the occupation number of that state and quantified by:

$$\eta(N^1) = N^{-1} e^{\beta g(N^1)} \tag{13.25}$$

where:

$$\beta = \ln\left(-\frac{y^1(t)-\overline{y}(t)}{y^0(t)-\overline{y}(t)}\right)\left(y^1(t)-y^0(t)\right)^{-1}$$

$$g(N^1) = -\frac{1}{2\beta}\frac{dH(N^1)}{dN^1} = -\frac{1}{2\beta}\ln\left(\frac{N^1}{N-N^1}\right)$$

$\overline{y}(t)$ is the average production $Y(t)/N$. The probability of being in state 1 in a given instant appears to be dependent on three factors: the number of firms already occupying the state, N^1; parameter β, which quantifies the impact on total output of the financial distress of the system; and function $g(N^1)$, which measures the average difference in the optimal levels of production. The situation and consequent behaviour of a firm is then linked to the state of the economy which, in turn, is determined by the number of firms that reduce output due to expected bankruptcy costs and by the level of this reduction, captured by $g(N^1)$. Equation (13.25) expresses the indirect interaction among firms and the feedback effects between macro-, meso- and micro-level.

13.4.3. Master equation solution: stochastic dynamics of trend and fluctuations

Applying the master equation solution (13.22) we can quantify and express in explicit form the stochastic dynamics of aggregate production and also decompose it into trend and cycle components. As a result, we can identify a long-run path dynamics (that eventually leads to a steady-state equilibrium, if it exists), and the fluctuations around this trend. In order to obtain such information, we assume that the fraction of firms in state 1 in a given moment is determined by its expected mean (m), the trend or *drift*, and, according to Aoki (2002), by an additive fluctuation component of order $N^{1/2}$ around this value, that is to say the *spread*:

$$N_k = Nm + \sqrt{N}s \tag{13.26}$$

As aggregate output is determined by the proportion of firms in the two states, the dynamics of N^1 describes the dynamics of production. Moreover, using equation (13.26), the solution of the master equation will return the expected long-run path of production and a probability distribution for business fluctuations.

Since an analytical solution for master equations can be obtained only under very specific and restrictive conditions (Risken, 1989), we can solve it using one of the approximation methods detailed in Aoki (2002) and Landini (2005) (see Appendix B). The asymptotically approximated solution of the master equation is given by the following system of coupled equations:

$$\frac{dm}{d\tau} = \lambda m - (\lambda + \gamma) m^2 \tag{13.27}$$

$$\frac{\partial Q}{\partial \tau} = \left[2(\lambda+\gamma)m - \lambda\right] \frac{\partial}{\partial s}(sQ(s)) + \frac{\left[\lambda m(1-m) + \gamma m^2\right]}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) \tag{13.28}$$

Equation (13.27) is a deterministic ordinary differential equation (the macroeconomic equation, in Aoki's words) which displays a logistic dynamics for the trend or drifting component. Equation (13.28) is a second-order stochastic partial differential equation, called *Fokker-Planck equation*, that drives the spreading component (that is the fluctuations around the trend) of the probability flow. Dynamics to the steady state value m^* is clearly convergent. Setting the l.h.s. of the macroscopic equation (13.27) to 0, we can calculate the steady-state value for m:

$$m^* = \frac{\lambda}{\lambda + \gamma} \tag{13.29}$$

Hence, directly integrating equation (13.27):

$$m(\tau) = \frac{\lambda}{(\lambda+\gamma) - ke^{-\psi\tau}} : \begin{cases} k = 1 - \dfrac{m^*}{m(0)} \\ \psi = \dfrac{(\lambda+\gamma)^2}{\lambda} \end{cases} \tag{13.30}$$

This differential equation describes the dynamics of fraction m of firms occupying state 1 at each point in time. It is fully dependent on transition rates. Solution of the equation for the spread component (see Appendix C) permits the distribution function θ to be calculated for spread s, thus determining the probability distribution of fluctuations:

$$\theta(s) = C\exp\left(-\frac{s^2}{2\sigma^2}\right) : \sigma^2 = \frac{\lambda\gamma}{(\lambda+\gamma)^2} = m^* \frac{\gamma}{\lambda+\gamma} \tag{13.31}$$

which looks like a Gaussian density. Fluctuations then also depend only on transition rates. Given the relationship between m and total production, the dynamics of our economy is now fully described since we have a differential equation for output dynamics, its equilibrium value, and a probability function for business fluctuations around the trend.

13.4.4. Output dynamics

Once the equilibrium distribution of the drift has been quantified, we are now able to obtain the steady-state aggregate production, Y^e:

$$Y^e = N\left[\frac{1}{r} - \frac{\lambda}{\lambda+\gamma}\frac{2c\mu}{r(r+2c\mu)}\right] = N\left[\frac{1}{r} - \frac{\lambda}{\lambda+\gamma}\left(y^0 - y^1\right)\right] \qquad (13.32)$$

As shown by equation (13.32), the dynamics depends on the difference among firms' optimal production levels and on the transition rates. The first component is given by the exogenous parameter c, that reflects institutional conditions, and by the probability of bankruptcy μ, that is the result of the relative financial condition of 'bad' firms. Indeed, the probability of demise depends on the distance between their equity ratio a^1 and the 'safety' level $\bar{a}$. The second component, recalling equations (13.1) and (13.16), is basically determined, on the one hand, by relative financial soundness or distress of firms of the two groups and, on the other, by the general financial situation of the system, revealed by the number of firms in each state. Substituting equations (13.16) and (13A.3) of Appendix A in equations (13.1), under detailed balance condition we can express the transition rates λ and λ as a function of the occupation numbers and shock price thresholds:

$$\lambda = \left(N^{-1}e^{\beta g(N^1)}\right)\left(2\bar{u}_\zeta - 1.5\right) \qquad (13.33)$$

$$\gamma = \left[1 - \left(N^{-1}e^{\beta g(N^1)}\right)\right]\left(-2.5\bar{u}_\iota + 1.5\right) \qquad (13.34)$$

Transition rates, hence the whole dynamics and final equilibrium of the system, are determined by a macro-factor, represented by η, and a micro factor, represented by probabilities ζ and ι. The joint effect of these variables gives rise to a mechanism that can be termed financial contagion. In particular, keeping in mind equations (13.15), the effect of the probability of bankruptcy at micro-level (reduction in optimal production q^1) appears to be amplified by the system's level of global financial distress. Distribution of agents, far from being neutral, is clearly the main determinant of the economic outcome. Moreover, since the system is governed by inherent uncertainty, all relationships have to be expressed as probability functions. Therefore the system dynamics appears to be fully stochastic, and the steady-state level of production can with difficulty be defined as a natural equilibrium.

13.5. SIMULATION RESULTS

To illustrate in greater detail the mechanism of the system and check the sensitivity of the results to variations in the parameters, we performed some simulations with Matlab software. The simulations highlighted the role of financial fragility in determining the evolution path for output and the model's capacity to generate multiple equilibria.

In order to get a treatable analytic approximation of mean-field interaction, equity ratios for firms in the two states are quantified by:

$$\begin{aligned} a^0(t) &= \overline{a} + \varepsilon\sigma(t) \\ a^1(t) &= \overline{a} - \varepsilon\sigma(t) \end{aligned} \tag{13.35}$$

where $0 < \varepsilon \leq 1$ and $\sigma(t)$ is the standard deviation in equity ratio distribution at time t. Parameter values are set to: $c = 0.3$, $\varepsilon = 0.1$.

The obtained series of m and of aggregate output are displayed in Figure 13.2. Trends are represented by continuous black lines, determined by means of equation (13.30). Fluctuations in n^1, which determine business cycles, follow the stochastic distribution expressed by equation (13.31). These two equations fully describe the economy's behaviour. The dynamics of m is convergent.

The final equilibrium percentage of 'bad' firms is positively related with interest rate. Indeed, a higher level of interest rate implies a higher value of m^* (and a lower level of steady state production), since it determines greater financing costs for firms, raising the probability of entering state 1 and lowering the probability of exit.

As demonstrated by equation (13.31), the percentage of heavily indebted firms also influences the amplitude of business cycles, amplifying the role of the risk of bankruptcy and then raising volatility and uncertainty. Figure 13.3 shows that a higher interest rate leads to a wider range of oscillation for output fluctuations due to the higher number of firms that occupy state 1. Indeed, the share n^1 constitutes a factor of uncertainty since the optimal level of output for 'bad' firms is subject to stochastic variations, due to the presence in their objective function of the probability of demise $\mu(t)$. Given equations (13.11) and (13.20) and considering the mean-field approximation (13.35), one can verify that firms' financial conditions, proxied by equity ratios, have a growing impact, as n^1 increases, not only on the level of output but also on its volatility. A higher level of financial distress in the system thus raises uncertainty, hence the higher risk involved in economic activity. The model appears suitable to provide indications for stabilization policies since the level of output and the amplitude of business fluctuations can be modified by handling the interest rates.

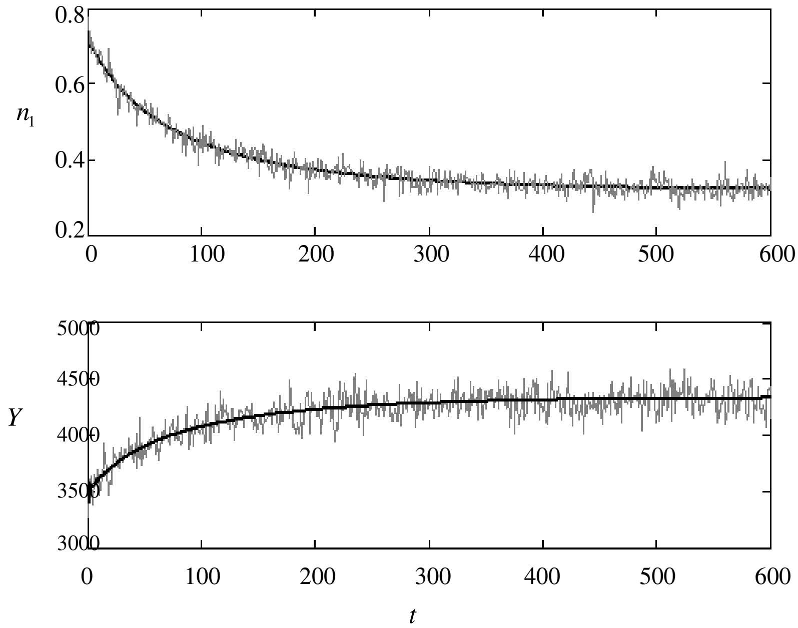

Figure 13.2. Trends and fluctuations for n^1 (upper panel) and aggregate production (lower panel)

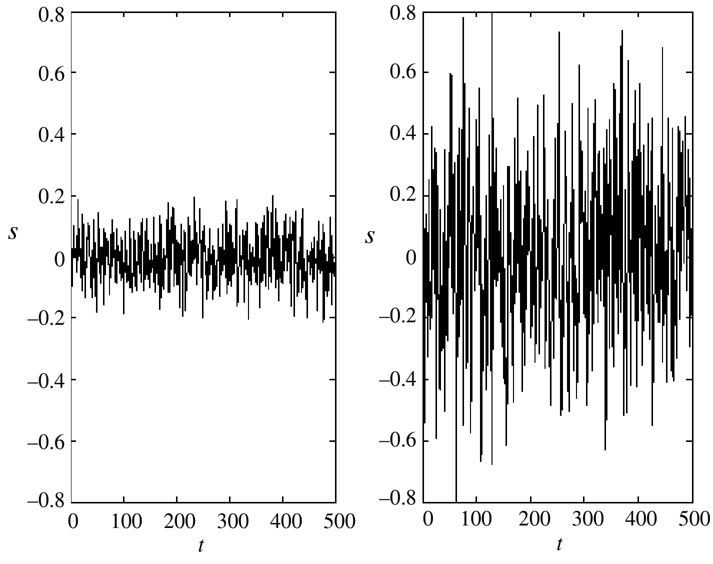

Figure 13.3. Spread for r = 0.1 (left panel) and r = 0.05 (right panel)

The positive correlation between financial fragility and uncertainty is confirmed by the bifurcation diagram (Figs. 13.4 and 13.5). For higher values of r, as ceiling effects on transition rates operate, the dynamics cannot be explosive. For interest rates below 10%, the system generates a chaotic dynamics. Figure 13.5 shows the detail of bifurcation diagrams for two different initial levels of m, showing that a higher starting value of $m(0)$ may result, also for an interest rate close to 0, in a higher m. Moreover, higher values of $m(0)$ will also generate a wider range of possible outcomes in the chaos region. Therefore, uncertainty about the effect of a variation in the interest rate turns out to be positively correlated to the financial fragility of the economy. As may be noted, although the underlying stochastic structure is rather simple, being structured in only two possible microstates, the model is able to generate complex dynamics with multiple equilibria.

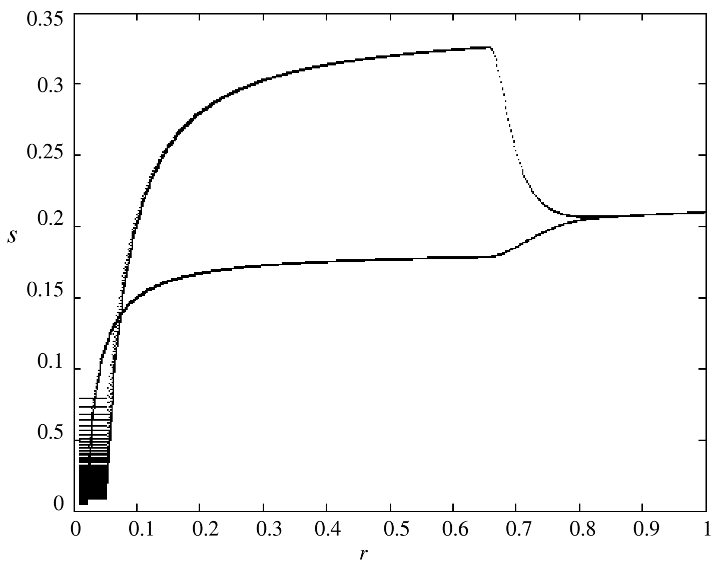

Figure 13.4. Bifurcation diagram for m *as a function of the interest rate* r

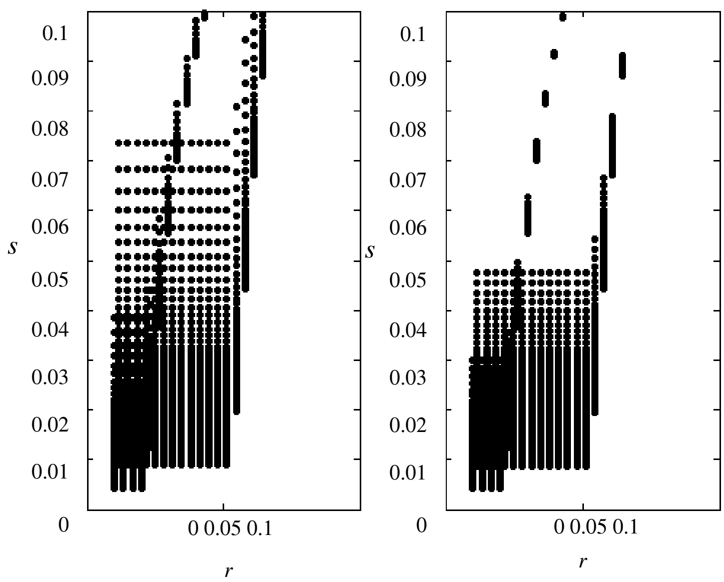

Figure 13.3. Spread for r = 0.1 (left panel) and r = 0.05 (right panel)

13.6. CONCLUDING REMARKS

Herein we developed an innovative framework for the aggregation of heterogeneous agents, originally conceived by Aoki, and applied it within a financial fragility model. The main result was endogenous generation of fluctuations of total production around a long-path trend, with growth paths and temporary variations dependent on firms' financial conditions. In particular, given a steady-state solution for the master equation, it is possible to identify an ordinary differential equation and a stochastic equation in order to describe, respectively, trend dynamics and fluctuation probability. The macro-level outcomes were obtained without formulating an *ad hoc* hypothesis about the dynamics of the system or its starting (or final) conditions. Importantly, the long-term trend can hardly be termed a natural level of production since, even though the proportions of firms in the two states asymptotically tend to a stationary value, this dynamics is purely stochastic.

Equilibrium probability formulations include micro- and macro-factors, permitting analytical representation and evaluation of the effect of indirect interaction among firms. The probability of a firm being forced to reduce production due to the risk of failure proves dependent also on the financial fragility of the other firms in the system, originating an effect that can be defined as stochastic financial contagion. Since aggregate production is itself dependent on the debt-to-equity ratio of each firm, this gives rise to feedback effects between micro- and macro-levels of the system. Micro-financial variables and aggregate production dynamics are then connected in functional relationships that allow detailed representation of transmission mechanisms within the system. At micro-level, the structure of optimization processes depends on firms' financial soundness, proving different for each type of firm. Modelling the links between financial fragility, business cycles and growth dynamics is then consistently microfounded, given the heterogeneity of firms' financial variables and the interaction among agents, and between agents and the macro-level of the system. This may open up interesting prospects for future developments of this stream of research.

This promising method, presented here through implementation in a financial fragility model, appears suited to application in any macroeconomic context where the presence and interaction of heterogeneous agents prevent the use of traditional tools for analytical identification of the system's emerging properties.

13A.1. APPENDIX A

We sketch here the basic steps for deriving the steady-state probability, referring the interested reader to the cited references. Stationary probability can be obtained by applying Brook's lemma which defines the local characteristic of continuous Markov chains. Hammersley and Clifford demonstrate that, under opportune conditions, for each Markov random field there is one, and only one, Gibbs random field, and define the functional form for the conjunct probability structure once the neighbourhood relations have been identified (Clifford, 1990). The expected stationary probability (eq:brook) of the Markovian process for N^1, when detailed balance holds, can then be expressed by:

$$P^e(N_k) \propto Z^{-1} e^{-\beta N U(N_k)} \tag{13A.1}$$

where $U(x)$ is the *Gibbs potential* and can be defined as a functional of the local dynamic characteristics of the state variable N_k. In particular we have:

$$e^{\beta g(N^1)} + e^{-\beta g(N^1)} = N \tag{13A.2}$$

Using the above formulation (Landini, 2005; Di Guilmi, 2008) the probability η can be expressed explicitly as a function of the state variable N^1:

$$\eta(N^1) = N^{-1} e^{\beta g(N^1)} \tag{13A.3}$$

where $g(N^1)$ is a function that evaluates the relative difference in the outcome as a function of N^1. β may be interpreted as an inverse measure of system uncertainty. The uncertainty among the different possible configurations in a stochastic system can be evaluated through a statistical entropy measure (Balian, 1991). Parameter β can be quantified by maximizing the statistical entropy of the system (Jaynes, 1957). In our case the problem is configured as follows:

$$\begin{cases} \max H(N^1,\ N^0) = -N^1(t)\ln(N^1(t)) - N^0(t)\ln(N^0(t)) \quad s.t. \\ N^1(t) + N^0(t) = N \\ N^1(t)y^1(t) + N^0(t)y^0(t) = Y(t) \end{cases} \tag{13A.4}$$

The first of the two constraints ensures the normalization of the probability function. The second ensures that all the wealth in the system is generated by firms in the two kinds of states. With suitable Lagrange multipliers equal to, respectively, $\delta_1 = 1 - \alpha$ and $\delta_2 = -\beta$, we get a closed solution for β:

$$\beta(t) = \ln\left(-\frac{y^1(t) - \bar{y}(t)}{y^0(t) - \bar{y}(t)}\right)\left(y^1(t) - y^0(t)\right)^{-1} \tag{13A.5}$$

Large values of β associated with positive values of $g(N^1)$ cause $\eta(N^1)$ to be larger than $1-\eta(N^1)$, making the transition from state 0 to state 1 more likely to occur than the opposite one. In binary models and for great N, the equation of the potential is:

$$U(N^j) = -2\int_0^{N^j} g(z)dz - \frac{1}{\beta}H(\underline{N})$$

where $H(\underline{N})$ is the Shannon entropy with $\underline{N} = (N^1, N^0)$. In order to determine the stationary points of probability dynamics we need to determine its peak (if it does exist). β is an inverse multiplicative factor for entropy: a relative high value of β means that the uncertainty in the system is low, with few firms exposed to bankruptcy risk. For values of β around 0, and higher volatility in the system, in order to determine the peak of probability dynamics we need to find the local minimum of the potential. Aoki (2002) shows that the points in which the potential is minimized are also the critical point of the aggregate dynamics of $P^e(N_k)$. Deriving the potential with respect to N^1 and then setting $U' = 0$:

$$g(N^1) = -\frac{1}{2\beta}\frac{dH}{dN^1} = -\frac{1}{2\beta}\ln\left(\frac{N^1}{N - N^1}\right) \tag{13A.6}$$

and using equation (13A.5), we get an explicit formulation for $g(N^1)$ in the stationary condition:

$$g(N^1) = \frac{y^0 - y^1}{2}$$

that quantifies the mean difference (for states) of the outcome.

13A.2. APPENDIX B

The master equation (13.22) must be modified, according to equation (13.26), and expressed as $\dot{Q}(s)$, a function of s:

$$\dot{P}(N_k) = \frac{\partial Q}{\partial t} + \frac{ds}{dt}\frac{\partial Q}{\partial s} = \dot{Q}(s) \tag{13A.7}$$

with transition rates reformulated in the following way:

$$b(s)=\lambda\left[N-Nm-\sqrt{N}s\right] \tag{13A.8}$$

$$d(s)=\gamma\left[Nm+\sqrt{N}s\right] \tag{13A.9}$$

Since

$$\frac{ds}{dt}=-N^{1/2}\frac{dm}{dt} \tag{13A.10}$$

equation (13A.7) can be expressed as:

$$\dot{Q}(s)=\frac{\partial Q}{\partial t}-N^{1/2}\frac{\partial Q}{\partial s}\dot{m} \tag{13A.11}$$

Now we rewrite again the master equation (13.22) and the transition rates by means of lead and lag operators. These operators make the two probability flows (in and out) homogeneous. Specifically the transition probabilities (13.1) become:

$$L[d(N_k)P(N_k,\ t)]=d(N_{k+1})P(N_{k+1}) \tag{13A.12}$$

$$L^{-1}[b(N_k)P(N_k,\ t)]=d(N_{k-1})P(N_{k-1}) \tag{13A.13}$$

so that the master equation will be expressed in this way:

$$\dot{Q}(s)=(L-1)[d(s)Q(s)]+(L^{-1}-1)[d(s)Q(s)] \tag{13A.14}$$

Using the modified transition rates (13A.12) and expanding the thus obtained master equation in inverse powers of s to the second order we get:

$$\begin{aligned}
&N^{-1}\tfrac{\partial Q}{\partial\tau}-N^{-1/2}\tfrac{dm}{d\tau}\tfrac{\partial Q}{\partial s}=\\
&N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[d(s)Q(s)]+N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[d(s)Q(s)]+\\
&-N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[b(s)Q(s)]+N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[b(s)Q(s)]+...\\
&=N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[(d(s)-b(s))Q(s)]+N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[(b(s)+d(s))Q(s)]+...
\end{aligned} \tag{13A.15}$$

where $\tau=t/N$. At this point, in order to match the component of the same orders of powers of N between equations (13A.7) and (13A.15), we need to rescale the variable $\tau=tN$. Knowing that:

$$d(s)-b(s)=(\lambda+\gamma)(Nm+\sqrt{N}s)-\lambda N=(\lambda+\gamma)N_k-\lambda N$$

$$d(s)+b(s)=(\lambda-\gamma)(Nm+\sqrt{N}s)+\lambda N=(\lambda-\gamma)N_k+\lambda N$$

and taking the derivatives up to the second order, it is possible to obtain what Aoki (2002) defines as *diffusion approximation*:

$$N^{-1}\frac{\partial Q}{\partial \tau}-N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s}=$$

$$(\lambda+\gamma)Q(s)+N^{-1/2}\left[d(s)-b(s)\right]\left(\frac{\partial}{\partial s}\right)\ Q(s)+N^{-1}\frac{1}{2}\left[b(s)+d(s)\right]\left(\frac{\partial}{\partial s}\right)Q(s) \tag{13A.16}$$

Applying the polynomial identity principle to equation (13A.16) for powers of N of order -1 we get a formulation for the *Fokker-Planck* equation:

$$\begin{aligned}N^{-1}\frac{\partial Q}{\partial t}&=-b'(m)s\frac{\partial Q}{\partial s}+b(m)\frac{1}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s)-b'(m)sQ(s)\\&+d'(m)s\left(\frac{\partial}{\partial s}\right)\ Q(s)+d(m)\frac{1}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s)-d'(m)Q(s)\\&=\left[d'(m)-b'(m)\right]\ \left(\frac{\partial}{\partial s}\right)\ Q(s)+\frac{1}{2}\left[d(m)+b(m)\right]\ \left(\frac{\partial}{\partial s}\right)^2 Q(s)\end{aligned} \tag{13A.17}$$

The generic asymptotic approximated solution for the master equation will then be given by the solution of the following coupled dynamic system of equations:

$$\begin{cases}\dfrac{dm}{d\tau}=\rho(m)\\[2ex]\dfrac{\partial Q}{\partial \tau}=-\rho'(m)\left(\dfrac{\partial}{\partial s}\right)\left[sQ(s)\right]+\dfrac{1}{2}\alpha(m)\left(\dfrac{\partial}{\partial s}\right)^2 Q(s)\\[2ex]s.t.\ \ \rho(m)=b(m)-d(m),\ \ \alpha(m)=b(m)+d(m)\end{cases} \tag{13A.18}$$

In order to arrive at an explicit solution we introduce a modification in the transition rates of equations (13.1), supposing that the probability η is equal to the observed frequency of firms occupying state 1. The new transition rates are then:

$$\begin{cases}b_n=r(N_k+1\,|\,N)=\zeta\dfrac{N_k}{N}\dfrac{N-N_k}{N}\\[2ex]d_n=l(N_k-1\,|\,N)=\iota\dfrac{N_k}{N}\dfrac{N_k-1}{N}\end{cases} \tag{13A.19}$$

where the factor N_k/N allows us to interpret, respectively, the probability transition ζ as a constant of proportionality between the birth rate per individual and the deviation from the upper bound $N-N_k$ and the probability transition ι as a constant of proportionality between the death rate per individual and the deviation from the lower bound or N_k-1. One can thus set the two functions:

$$\begin{aligned}\lambda(N_k)&=\lambda\frac{N_k}{N}\\ \gamma(N_k)&=\gamma\frac{N_k}{N}\end{aligned}\tag{13A.20}$$

Substituting the transition rates (13A.20) in the master equation (13.22) and collecting terms with λ and γ, after some simple but tedious algebraic passages, we obtain:

$$\begin{aligned}\frac{dP}{dt}&=N^{-2}\{\gamma[N_k(N_k+1)L(P)+2nP]+\\ &-\lambda\left[(N_k-1)(N-N_k+1)L^{-1}(P)+(N-2n+1)P\right]\}\end{aligned}\tag{13A.21}$$

where $L(P)$ and $L^{-1}(P)$ are lead and lag operators, reformulated in the following way according to Aoki (1996) and Landini and Uberti (2008):

$$L(P)=\sum_{z=1}^{\infty}\frac{N^{-z/2}}{z!}\left(\frac{\partial}{\partial s}\right)^{z}Q(s)\tag{13A.22}$$

$$L^{-1}(P)=\sum_{z=1}^{\infty}\frac{(-)^{z}N^{-z/2}}{z!}\left(\frac{\partial}{\partial s}\right)^{z}Q(s)\tag{13A.23}$$

Substituting the above indicated operators into equation (13A.21), it becomes:

$$\begin{aligned}\frac{dP}{dt}&=N^{-2}\left\{\textstyle\sum_{z=1}^{\infty}\left[D(N_k)+(-)^{z}B(N_k)\right]\ \frac{N^{-z/2}}{z!}\left(\frac{\partial}{\partial s}\right)^{z}Q(s)\right\}+\\ &+N^{-2}\{[2\gamma N_k-\lambda(N-2n+1)]\ Q(s)\}\end{aligned}\tag{13A.24}$$

where:

$$\begin{cases}B(N_k)=\lambda(N_k-1)(N-N_k+1)=\lambda N(N_k-1)-\lambda(N_k-1)^2=B(m)\\ D(N_k)=\gamma N_k(N_k+1)=D(m)\end{cases}\tag{13A.25}$$

The specification of the drift displayed in equation (13.26) implies that:

$$\begin{cases} N_{k+1} = Nm + \sqrt{N}(s + N^{-1/2}) \\ N_k = Nm + \sqrt{N}s \\ N_{k-1} = Nm + \sqrt{N}(s - N^{-1/2}) \end{cases} \tag{13A.26}$$

Using these specifications in equation (13.26), we obtain:

$$\begin{cases} B(m) = \lambda\left[N^2 m(1-m) + N^{3/2}s(1-2m) + N(2m - s^2 - 1) + 2N^{1/2}s - 1\right] \\ D(m) = \gamma\left[N^2 m^2 + N^{3/2}2ms + N(m + s^2) + N^{1/2}s\right] \end{cases} \tag{13A.27}$$

Now, expanding to the second order equation approximation equation (13A.24), that is to say for $z=1,\ 2,$ we get:

$$\begin{cases} z=1: \quad N^{-1/2}\left[D(m) - B(m)\right] = \\ N^{3/2}\left[-\lambda m(1-m) + \gamma m^2\right] + N\left[2ms(\lambda+\gamma) - \lambda\langle s\rangle\langle s\rangle\right] + \\ +N^{1/2}\left[s^2(\lambda+\gamma) + (\gamma - 2\lambda)m + 1\lambda\right] - 2\lambda s + N^{-1/2}(\gamma s - \lambda) \\ z=2: \quad \frac{N^{-1}}{2}\left[D(m) + B(m)\right] = \\ N\left[\lambda m(1-m) + \gamma m^2\right] + N^{1/2}\left[2ms(\lambda+\gamma) - \lambda s\right] + \\ +\left[s^2(\gamma-\lambda) + m(\gamma + 2\lambda) - \lambda\right] + N^{-1/2}2\lambda s + N^{-1}(\gamma s + \lambda) \end{cases} \tag{13A.28}$$

and substituting it into equation (13A.25), it gives back the following approximated master equation:

$$\begin{aligned} \frac{dP}{dt} = \; & \left\{N^{-1/2}\left[-\lambda m(1-m) + \gamma m^2\right] + N^{-1}\left[2ms(\lambda+\gamma) - \lambda s\right]\right\} \frac{\partial}{\partial s}Q(s) + \\ & +\left\{-N^{-3/2}\left[s^2(\lambda+\gamma) + (\gamma - 2\lambda)m + \lambda\right] - N^{-2}2\lambda s + \right. \\ & \left. -N^{-5/2}(\gamma s - \lambda)\right\} \frac{\partial}{\partial s}Q(s) + \frac{1}{2}\left\{N^{-1}\left[\lambda m(1-m) + \gamma m^2\right] + \right. \end{aligned}$$

$$+N^{-3/2}\left[2ms(\lambda+\gamma)-\lambda s\right]\Big\}\left(\frac{\partial}{\partial s}\right)^2 Q(s)+$$

$$+\frac{1}{2}\Big\{N^{-2}\left[s^2(\gamma-\lambda)+m(\gamma+2\lambda)-\lambda\right]+$$

$$+N^{-5/2}\lambda s+N^{-3}(\gamma s+\lambda)\Big\}\left(\frac{\partial}{\partial s}\right)^2 Q(s)+$$

$$+\left\{N^{-1}\left[2m(\lambda+\gamma)-\lambda\right]+N^{-3/2}\left[2s(\lambda+\gamma)\right]-N^{-2}\lambda\right\} \tag{13A.29}$$

Given that:

$$\frac{dP}{dt}=\frac{\partial Q}{\partial t}-N^{-1/2}\frac{dm}{dt}\frac{\partial Q}{\partial s} \tag{13A.30}$$

in order to match the higher order terms in powers of N, we have to rescale time as $t=N\tau$:

$$\frac{dP}{dt}=\frac{\partial Q}{\partial t}-N^{-1/2}\frac{dm}{dt}\frac{\partial Q}{\partial s}\Leftrightarrow N^{-1}\frac{dP}{d\tau}=N^{-1}\frac{\partial Q}{\partial \tau}-N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s} \tag{13A.31}$$

Then we have to equal the two formulations thus obtained for the master equation: equation (13A.29) and equation (13A.31). This can be done by matching the terms that have the same power of N. Then we collect terms of order N^{-1} in equation (13A.29) so as to match them with $\partial Q/\partial\tau$ of equation (13A.31) and those of order $N^{-1/2}$ to set them equal to $N^{-1/2}\dot{m}(\partial Q/\partial s)$. All the other terms asymptotically vanish as $N\to\infty$. In this way we get:

$$-N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s}=-N^{-1/2}\left[\lambda m(1-m)-\gamma m^2\right]\frac{\partial}{\partial s}Q(s) \tag{13A.32}$$

$$\begin{aligned}N^{-1}\frac{\partial Q}{\partial \tau}= \quad & N^{-1}\left[2m(\lambda+\gamma)-\lambda\right]\frac{\partial}{\partial s}\left[sQ(s)\right]+\\ & +\frac{N^{-1}}{2}\left[\lambda m(1-m)+\gamma m^2\right]\left(\frac{\partial}{\partial s}\right)^2 Q(s)\end{aligned} \tag{13A.33}$$

Asymptotically approximated solution of master equation is given by the following system of coupled equations:

$$\frac{dm}{d\tau}=\lambda m-(\lambda+\gamma)m^2 \tag{13A.34}$$

$$\frac{\partial Q}{\partial \tau} = [2(\lambda+\gamma)m-\lambda]\ \frac{\partial}{\partial s}(sQ(s)) + \frac{\left[\lambda m(1-m)+\gamma m^2\right]}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) \quad (13A.35)$$

13A.3. APPENDIX C

Below we determine a solution for the Fokker-Planck equation in terms of $Q(s)$. Using $\theta(s)$ to indicate the stationary probability for $Q(s)$ and setting the equilibrium condition $\dot{Q}=0$ (that implies $\dot{\theta}=0$), it is possible to obtain:

$$-\left[2(\lambda+\gamma)m^* - \lambda\right]\ s\theta(s) = \frac{\left[\lambda m^*(1-m^*)+\gamma m^{*2}\right]}{2}\left(\frac{\partial}{\partial s}\right)\ \theta(s) \quad (13A.36)$$

Rewriting (13A.36) more conveniently as:

$$\frac{2\left[\lambda - 2(\lambda+\gamma)m^*\right]}{\left[\lambda m^*(1-m^*)+\gamma m^{*2}\right]}s = \frac{1}{\theta(s)}\left(\frac{\partial}{\partial s}\right)\ \theta(s) \quad (13A.37)$$

and integrating it with respect to s, we obtain:

$$\log\theta(s) = C + \frac{\lambda - 2(\lambda+\gamma)m^*}{\lambda m^*(1-m^*)+\gamma m^{*2}}s^2$$

$$\Updownarrow \quad (13A.38)$$

$$\theta(s) = C\exp\left(\frac{\lambda - 2(\lambda+\gamma)m^*}{\lambda m^* + (\gamma-\lambda)m^{*2}}s^2\right)$$

Then, substituting $m^* = \lambda/(\lambda+\gamma)$, we get the final result:

$$\theta(s) = C\exp\left(-\frac{s^2}{2\sigma^2}\right) : \sigma^2 = \frac{\lambda\gamma}{(\lambda+\gamma)^2} \quad 13A.39$$

NOTES

1. The probabilistic dynamics shown here shares some features with the compartmental models. For a reference see Bischi (1998).
2. Consequently, negative investment implies negative variation of debt, that is disinvestment is used to repay debt.
3. We can now quantify the upper bound of the demand of money. Given the optimal levels of capital for each cluster of firms, k^1 and k^0 , the quantity of demanded credit reaches its maximum when a^1 and a^0 reach their minimum. a^1 cannot go below $r-2.5/q^1$, at which

value μ becomes equal to 1. The minimum level for a^0 is, by definition, $\overline{a} = r - 1.5/q^1$. For these values we have:

$$\lim_{a^0 \to \overline{a}} B^0 = \frac{(q^0)^2}{2}(1 - r + 1.5/q^1)$$

$$\lim_{a^1 \to \overline{a}} B^1 = \frac{(q^1)^2}{2}(1 - r + 2.5/q^1)$$

Consequently, the maximum level of the demand of debt $B = B^0 + B^1$ is:

$$\max(B) = N^0\left[\frac{(q^0)^2}{2}\left(1 - r + \frac{1.5}{q^1}\right)\right] + N^1\left[\frac{(q^1)^2}{2}\left(1 - r + \frac{2.5}{q^1}\right)\right]$$

that cannot grow indefinitely since $q^0 = 1/r$ and $q^1 < q^0$ as demonstrated below.

4. For details on derivation see Aoki (2002, ch. 3), Landini (2005, p. 252) and Kelly (1979).

REFERENCES

Aoki, M. (1996), *New Approaches to Macroeconomic Modeling*, Cambridge University Press.

Aoki, M. (2002), *Modeling Aggregate Behavior and Fluctuations in Economics*, Cambridge: Cambridge University Press.

Aoki, M. and H. Yoshikawa (2006), *Reconstructing Macroeconomics*, Cambridge: Cambridge University Press.

Axelrod, R. (1997), 'Advancing the Art of Simulation in the Social Sciences', *Santa Fe Institute Working Papers*, no. 97-05-048.

Axtell, R., R. Axelrod, J.M. Epstein and M.D. Cohen (1996), 'Aligning Simulation Models: A Case Study and Results', *Computational & Mathematical Organization Theory*, **1**(2), 123–41.

Balian, R. (1991), *From Microphysics to Macrophysics*, Volume I, Berlin/Heidelberg/New York: Springer-Verlag.

Bischi, G.I. (1999), 'Compartmental Analysis of Economic Systems with Heterogeneous Agents: An Introduction', in A. Kirman and M. Gallegati (eds), *Beyond the Representative Agent*, Cheltenham, UK: Edward Elgar, pp. 181–214.

Brook, D. (1964), 'On the Distinction between the Conditional Probability and the Joint Probability Approaches in the Specification of Nearest-Neighbour Systems', *Biometrica*, **51**(3–4), 481–83.

Clifford, P. (1990), 'Markov Random Fields in Statistics', in G.R. Grimmet and D.J.A. Welsh (eds), *Disorder in Physical Systems. A Volume in Honour of John M. Hammersley*, Oxford: Clarendon Press, pp 19–32.

Delli Gatti, D., C. Di Guilmi, E. Gaffeo, G. Giulioni, M. Gallegati and A. Palestrini, (2005), 'A New Approach to Business Fluctuations: Heterogeneous Interacting Agents, Scaling Laws and Financial Fragility', *Journal of Economic Behavior and Organization*, **56**(4), 489–512.

Di Guilmi, C. (2008). *The Stochastic Dynamics of Business Cycles: Financial Fragility and Mean-Field Interaction*, Frankfurt am Main: Peter Lang Publishing Group.

Forni, M. and M. Lippi (1997), *Aggregation and the Microfoundations of Dynamic Macroeconomics*, Oxford: Oxford University Press.

Greenwald, B. and J.E. Stiglitz (1990), 'Macroeconomic Models with Equity and Credit Rationing', in R. Hubbard (ed.), *Information, Capital Markets and Investment*, Chicago: Chicago University Press.

Greenwald, B. and J.E. Stiglitz (1993). 'Financial Market Imperfections and Business Cycles', *Quarterly Journal of Economics*, **108**(1), 77–114.

Hinich, M.J., J. Foster and P. Wild (2006), 'Structural Change in Macroeconomic Time Series: A Complex Systems Perspective', *Journal of Macroeconomics*, **28**(1), 136–150.

Jaynes, E. T. (1957), 'Information Theory and Statistical Mechanics', Physical Review, **106**(4), 620–30.

Kelly, F. (1979), *Reversibility and Stochastic Networks*, New York: John Wiley and Sons Ltd.

Kirman, A.P. (1992), 'Whom or What Does the Representative Individual Represent?, *Journal of Economic Perspectives*, **6**(2), 117–36.

Landini, S. (2005), *Modellizzazione stocastica di grandezze economiche con un approccio econofisico*, PhD thesis, University Bicocca, Milan.

Landini, S. and M. Uberti (2008), 'A Statistical Mechanic View of Macrodynamics in Economics', *Computational Economics*, **32**(1), 121–146.

Minsky, H. (1963), 'Can 'it' Happen Again?', in D. Carson (ed.), *Banking and Monetary Studies*, Homewood: Richard Irwin, Reprinted in H. Minsky, *Inflation, Recession and Economic Policy*, New York: ME Sharpe, 1982.

Opper, M. and D. Saad (2001), *Advanced Mean Field Methods: Theory and Practice*. Cambridge, MA: The MIT Press.

Risken, H. (1989), *Fokker-Planck Equation. Method of Solutions and Applications*, Berlin: Springer Verlag.

Rosser, J.B. (ed.), (2004), *Complexity in Economics*, Northampton, MA: Edward Elgar.

Woess, W. (1996), *Catene di Markov e teoria del potenziale discreto*, Quaderni UMI no. 41, Bologna: Pitagora Editrice.